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Renormalization Conditions and the Sliding Scale in the Implicit Regularization Scheme: A Simple Connection

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We describe in detail how a sliding scale is introduced in the renormalization of a QFT according to integer-dimensional implicit regularization scheme. We show that since no regulator needs to be specified at intermediate steps of the calculation, the introduction of a mass scale is a direct consequence of a set of renormalization conditions. As an illustration the one-loop β-function for QED and $\lambda\varphi^4$ theories are derived. They are given in terms of derivatives of appropriately systematized functions (related to definite parts of the amplitudes) with respect to a mass scale μ. Our formal scheme can be easily generalized for higher loop calculations.

KEY WORDS: renormalization group; regularization methods.

1. INTRODUCTION

In dealing with quantum field theories usually divergent quantities (Green functions, Feynman integrals, etc.) are found in some large energy region. These large momenta correspond to short distance singularities resulting from badly defined quantities such as products of fields at the same point. As these elementary Green functions are not well-defined functions but rather distributions and since product of distributions is ill-defined, this leads to the divergences in large momenta mentioned above. These quantities should, in principle, have physical content. In order to proceed, a regulator must be imposed in the divergent expressions and, in the process of renormalization, finite parameters are defined. The renormalization program has to be a systematic and unambiguously fixed algorithm that satisfies

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the fundamental properties of locality and causality (de Witt and Stora, 1971): it should correspond to the addition of local counterterms to the Lagrangian density. The predictivity of the theory, that is the ability to obtain results valid to all orders of perturbation theory such as the renormalization group equation, relies on these logical conditions. In general, any renormalization procedure involves two steps (Bonneau, 1990):

1. *A regularization followed by a subtraction procedure*;
2. *A set of renormalization conditions in order to define the parameters of the theory.*

Step 1 refers to a systematic, uniquely fixed and consistent procedure, powerful enough to investigate to all orders of perturbation theory its renormalizability, fields, finite parameters, and symmetries. A common feature of regularization schemes is the introduction of (at least) one regularizing parameter (sharp cutoff Λ^2, Pauli–Villars masses m_i^2, dimension of space–time $D = 4 - \epsilon, \ldots$), and in the process of subtracting the divergences, the resulting expressions will remain finite when the regulator is suppressed. Stated generally, the introduction of regulators is followed by the introduction of mass parameters. The second step refers to the task of defining the parameters of the theory (fields, masses, couplings) in a suitable energy scale μ in each order of perturbation (Peskin and Schroeder, 1995). This is accomplished by subtracting each primitive divergence from a specific parameter. In other words, after introduction of a regularization, the coefficients of the counterterms are completely determined by renormalization conditions, imposed order by order on the primitively divergent Green functions. The infinite subtraction is performed in an energy scale μ, and the study of the behavior of renormalized Green functions with μ is an important branch in particle physics, issuing the renormalization group techniques (Brizola *et al.*, 1999; Callan, 1970; Callan *et al.*, 1970; Gell-mann and Low, 1954; Stueckenberg and Peterman, 1953; Symanzik, 1970).

In implementing Step 1 mentioned above, the most successful and popular regularization procedure is the dimensional regularization (DR) ('t Hooft and Veltman, 1972). The great success of DR is mainly due to the fact that it automatically respects gauge invariance. It is known, however, that it presents problems in dimensional-dependent theories like chiral or supersymmetric theories. The proposed alternative, dimensional reduction (Siegel, 1979) is usually employed in these cases, although inconsistencies may arise at high orders (Jack and Jones, in press; Siegel, 1980). In this context it is most desirable to develop other regularization schemes, specific to four dimensions which preserves the consistency of DR. Recently two such schemes were proposed, the *differential regularization* and the *implicit regularization*. The first one is established in coordinate space and the latter in momentum space. A mass scale is automatically introduced in differential regularization, for dimensional reasons, when the regulated propagators

are defined. The relation between this mass parameter and the choice of the renormalization point, as in DR, is not direct. As we will show, since in the implicit regularization scheme no specific form of a regulator needs to be specified, the calculation is not contaminated by regularization parameters in any step. This technique is therefore most adequate to establish, in a regularization independent way, the relation between the mass scale and the choice of the renormalization point. This is the main purpose of the present contribution.

In Section 2 we introduce the implicit regularization technique, the sliding scale in the renormalization procedure, and systematize the finite contributions of two- and three-point functions. In Section 3 we discuss the renormalization group within our scheme and derive the β-function for QED and $\lambda\varphi^4$. Relation between our approach and the other current schemes can be found in Section 4. In Section 5 we compare differential, dimensional, and implicit renormalizations. Final remarks are in Section 6.

2. THE IMPLICIT REGULARIZATION TECHNIQUE AND FINITE CONTENT OF ONE-LOOP AMPLITUDES

In this section we define the implicit regularization technique (IRT) for a general n-loop calculation. We closely follow Gobira and Nemes (submitted). The first step in implementing the IRT is to assume an implicit regularization[4] whenever a divergence occurs in a Feynman integral. After taking the Dirac trace (if required), one identifies the divergence degree of the integrals and manipulates the integrand by means of algebraic identities until the external momenta-dependent parts are isolated solely in terms of finite contributions. To separate the divergences the following identity will be used recursively until the last term acquires a negative degree of divergence in an integration over k in four space–time dimensions:

$$\frac{1}{[(k+p)^2-m^2]} = \sum_{j=0}^{N} \frac{(-1)^j (p^2+2pk)^j}{(k^2-m^2)^{j+1}} + \frac{(-1)^{N+1}(p^2+2pk)^{N+1}}{(k^2-m^2)^{N+1}} \frac{1}{[(k+p)^2-m^2]}. \qquad (1)$$

By convenience we divide the diagrams which contribute to a given order in two classes: the first which do not contain diagrams which possess two point functions as subdivergences and in the second class those which do.

Let us start with the first class of diagrams. To show how the procedure works it is enough to consider a general Feynman amplitude with one external momentum p, one coupling constant λ, and one mass parameter m. We work in the

[4] The only required condition about the implicit regularization is that it must be even in the loop momenta and with a connection limit that returns the original integrand.

four-dimensional space–time although the generalization to any integer dimension is straightforward. We denote by q a sum of internal momenta k_i. The amplitude in question can always be written as

$$\Gamma = \prod_{i=1}^{n} \int_{\Lambda} \frac{d^4 k_i}{(2\pi)^4} R(p, q, m, \lambda) \left[\prod_{j=1}^{l} f_j(p, q_j, m^2) \right], \tag{2}$$

where Γ represents $1 - PI$ diagrams,

$$f_j(p, q_j, m^2) = \frac{1}{[(p - q_j)^2 - m^2]}, \tag{3}$$

and

$$l = \text{number of } f \text{ structures}$$

$$n = \text{number of loops.}$$

Note that we have explicitly separated the terms involving the external momentum in the denominator, from which nonlocal divergent contributions can arise after integration over the internal momenta. The structure $R(p, q, m, \lambda)$ contains all other ingredients of the amplitude, such as coupling constants and results of Dirac traces.

For simplicity we adopt the following notation:

$$\Gamma = (\Pi R)(\Pi f), \tag{4}$$

where

$$(\Pi R) = \prod_{i=1}^{n} \int_{\Lambda} \frac{d^4 k_i}{(2\pi)^4} R(p, q, m, \lambda) \tag{5}$$

and

$$(\Pi f) \prod_{j=1}^{l} f_j(p, q_j, m^2). \tag{6}$$

As discussed before, the source of all possible troubles in the renormalization process will arise from the structure (Πf). Our method focus attention on these structures. In order to clearly separate finite, "trivial" divergences (whose dependence on the external momenta is only a polynomial) from the nonlocal divergences we use a strategy which is completely based on the identity (1).

Define the operator T^D which acts on each structure f in the following way:

$$T^0 f = \frac{1}{q_j^2 - m^2} + \frac{2pq_j - p^2}{(q_j^2 - m^2)} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\}, \tag{7}$$

$$T^1 f = \frac{1}{q_j^2 - m^2} + \frac{(2pq_j - p^2)}{\left(q_j^2 - m^2\right)^2} + \frac{(2pq_j - p^2)^2}{\left(q_j^2 - m^2\right)^2} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\}, \quad (8)$$

$$T^2 f = \frac{1}{q_j^2 - m^2} + \frac{(2pq_j - p^2)}{\left(q_j^2 - m^2\right)^2} + \frac{(2pq_j - p^2)^2}{\left(q_j^2 - m^2\right)^3} + \frac{(2pq_j - p^2)^3}{\left(q_j^2 - m^2\right)^3} \left\{ \frac{1}{[(p - q_j)^2 - m^2]} \right\}. \quad (9)$$

Note that the action of the operator T^D is equivalent to a Taylor expansion around zero external momentum, where the first term is kept and the rest of the series is resumed, yielding thus a convenient identity. Note also that the degree of divergence of the various terms is decreasing.

The procedure we have in mind consists of applying the operation, in a particular amplitude with the superficial degree of divergence D, to *each* function f_j:

$$T^D \Gamma = (\Pi R) \prod_{j=1}^{l} T_j^D f_1(p, q_j, m^2). \quad (10)$$

The result of the operation will always have the form

$$T^D f(p, q, m^2) = f^{\text{div}}(p, q, m^2) + f^{\text{fin}}(p, q, m^2). \quad (11)$$

We define

$$f^{\text{div}}(p, q, m^2) = \sum_{i=0}^{D} f^i(p, q, m^2). \quad (12)$$

Let us exemplify. Take a quadratically divergent amplitude. To each contribution of the form

$$\frac{1}{(p - q_j)^2 - m^2},$$

we associate

$$f^0(q, m^2) = \frac{1}{q^2 - m^2}, \quad (13)$$

$$f^1(p, q, m^2) = \frac{2pq - p^2}{(q^2 - m^2)^2}, \quad (14)$$

$$f^2(p, q, m^2) = \frac{(2pq)^2}{(q^2 - m^2)^3}, \quad (15)$$

and

$$f^{\text{fin}}(p,q,m^2)=\frac{p^4-4p^2(pq)}{(q^2-m^2)^3}+\frac{(2pq-p^2)^3}{(q^2-m^2)^3[(p-q_j)^2-m^2]}. \tag{16}$$

The definitions (13)–(16) are not unique. It is simply convenient for our purposes. Using these we rewrite the amplitude as a sum of various contributions. According to our notation

$$T^D\Gamma=(\Pi R)\prod_{j=1}^{l}\left[\, f_j^{\text{div}}(p,q,m^2)+\, f_j^{\text{fin}}(p,q,m^2)\right]. \tag{17}$$

In this way we can identify three distinct contributions for the amplitude

$$T^D\Gamma=\Gamma^1_{\text{fin}}+\Gamma_{\text{local}}+\Gamma_{\text{nonlocal}}, \tag{18}$$

where

$$\Gamma^1_{\text{fin}}=(\Pi R)\prod_{j=1}^{l} f_j^{\text{fin}}(p,q,m^2). \tag{19}$$

The second contribution contains only local divergences and, for some particular (ΠR) structure, it can contain finite contributions too. It is identified as

$$\begin{aligned}\Gamma_{\text{local}}&=(\Pi R)\prod_{j=1}^{1} f_j^{\text{div}}(p,q,m^2).\\ &=\Gamma^2_{\text{fin}}+\Gamma^{\text{div}}_{\text{local}}.\end{aligned} \tag{20}$$

These local divergences correspond to counterterms which are characteristic of the order we are renormalizing. For example, they can have the form

$$\int_\Lambda \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}+p^2 I_{\log}(m^2)+\text{finite part}. \tag{21}$$

The last term in Eq (18), namely the cross-terms, contains finite contributions as well as "nonlocal" divergences.

$$\Gamma_{\text{nonlocal}}=\Gamma^3_{\text{fin}}+\Gamma^{\text{div}}_{\text{nonlocal}}. \tag{22}$$

These nonlocal divergence contributions will always appear because of the divergent subdiagrams (beyond two-point functions) contained in the graph. As we will show next in a particular example, the renormalization of previous orders will always allow one to cancel these contributions if the theory is renormalizable. In the present scheme the result is automatic and follows from the operation we have just defined, in an algebraic manner. There is no need for graphic representations of relevant contributions, although it is possible.

The renormalized amplitude say, in nth-loop order, can therefore be defined as

$$\begin{aligned}\Gamma_{\mathrm{R}}^{(n)} &= T^{D}\Gamma^{(n)} - \Gamma_{\mathrm{local}}^{\mathrm{div}(n)} - \Gamma_{\mathrm{nonlocal}}^{\mathrm{div}(n)} \\ &= \Gamma_{\mathrm{fin}}^{1(n)} + \Gamma_{\mathrm{fin}}^{2(n)} + \Gamma_{\mathrm{fin}}^{3(n)},\end{aligned} \tag{23}$$

where the contributions $\Gamma_{\mathrm{local}}^{\mathrm{div}(n)}$ and $\Gamma_{\mathrm{nonlocal}}^{\mathrm{div}(n)}$ contain the counterterms typical of order n as well as the counterterms coming from divergent subdiagrams of previous order. Notice from the above equation that our framework automatically delivers the counterterms

$$\Gamma_{\mathrm{CT}}^{1} = -\Gamma_{\mathrm{local}}^{\mathrm{div}} - \Gamma_{\mathrm{nonlocal}}^{\mathrm{div}}, \tag{24}$$

and just as in Bogolubov Parasiuk Hepp Zimmermann, by subtracting off the necessary counterterms leaves with the finite part of the amplitude, the main difference being that here the counterterms can be read out of the procedure.

Now we proceed to the second class of diagrams, namely those which contain two-point functions as subdiagrams. Let us call U all the two-point diagrams contained in a given amplitude Γ. It is easy to see that they can be factored out inside of the total amplitude in the following sense:

$$\Gamma = \prod_{\text{all } \Sigma_j \in U} \mathcal{R}_j \Sigma_j^{(l)}(q_j^2), \tag{25}$$

where $\mathcal{R}_j$ stands for the remaining pieces in the amplitude, j characterizes a specific two-point function, which is one of the integration momenta (external to Σ_j). Now since the operation $T^{D}\Gamma$ is an identity, i.e., $T^{D}\Gamma = \Gamma$, we can define the partially renormalized amplitude (with all two-point function subdiagrams properly renormalized) as follows:

$$\bar{\Gamma} = \Gamma + \Gamma_{\mathrm{CT}}^{2}. \tag{26}$$

Therefore we have

$$\Gamma_{\mathrm{CT}}^{2} = \prod_{\text{all } \Sigma_j \in U} \mathcal{R}_j\left[\delta_j^{(l)} m^2 - A_j^{(l)} q_j^2\right] \tag{27}$$

and Γ_{CT}^{2} are all counterterms characteristic subdiagrams involving two-point functions. $\delta_j^{(l)} m^2$ stands for the mass renormalization and $A_j^{(l)}$ for the wave function renormalization. In order to get the renormalized amplitude of order n from $\bar{\Gamma}$ one proceeds in the same way as for diagrams of class one defined above. We thus have

$$\begin{aligned}\Gamma_{\mathrm{R}} &= T^{D}\bar{\Gamma} - \bar{\Gamma}_{\mathrm{local}}^{\mathrm{div}} - \bar{\Gamma}_{\mathrm{nonlocal}}^{\mathrm{div}} \\ &= \bar{\Gamma}_{\mathrm{fin}}^{1} + \bar{\Gamma}_{\mathrm{fin}}^{2} + \bar{\Gamma}_{\mathrm{fin}}^{3}.\end{aligned} \tag{28}$$

Summarizing, the amplitudes will be written as the sum of basically divergent parts (defined in each order of perturbation), terms containing differences between

divergent integrals of the same degree of divergence (which we will call consistency relations), and finite parts. A word about the consistency relations is in order: An important ingredient of the IRT are the so-called consistency relations expressed by differences between divergent integrals of the same degree of divergence. It was shown (Battistel *et al.*, 1998) that such consistency relations should vanish in order to avoid ambiguities related to the various possible choices for the momentum routing in certain amplitudes involving loops, consistently with gauge invariance. This is an important feature of dimensional regularization and it can be easily checked that the consistency relations are readily fulfilled in the framework of dimensional regularization. Alternatively and more generically we can assign an arbitrary value to such consistency relations and let general symmetry properties of the theory or physical constraints determine their value (Scarpelli *et al.*, 2001).

Let us now consider the massive $\lambda\varphi_4^4$ theory (Weinberg, 1996a)

$$\mathcal{L}_B = \frac{1}{2}\partial_\nu\varphi_B\partial^\nu\varphi_B - \frac{m_B^2}{2}\varphi_B^2 - \frac{\lambda_B}{4!}\varphi_B^4. \tag{29}$$

The index B means bare parameters. In order to renormalize the theory the multiplicative renormalization constants z_φ, z_λ, z_m are introduced:

$$\varphi_B = z_\varphi^{1/2}\varphi, \tag{30}$$

$$\lambda_B = z_\varphi^{-2} z_\lambda \lambda, \tag{31}$$

$$m_B^2 = z_\varphi^{-1} z_m m^2. \tag{32}$$

Perturbative calculations yield an expansion of n-point Green's function $\Gamma^{(n)}$ in a *conventional defined coupling*,

$$\Gamma^{(n)}(p^2) = \sum_{i=0}^{\infty} c_i^{(n)} \lambda^i, \tag{33}$$

where λ is finite and defined in a conventional renormalization point. Let us define the conventional coupling via the renormalization conditions:

$$\Gamma^{(2)}(p^2) = m^2 \quad \text{at } p^2 = 0, \tag{34}$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\lambda \tag{35}$$

$$\text{at } p_i^2 = 0 \text{ and } p_i p_j = 0, \tag{36}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}(p^2) = 1 \quad \text{at } p^2 = 0. \tag{37}$$

The choice of this particular value of the external momenta in (34), (35), and (37) was guided only for convenience, especially since it renders simple expressions. But it is worthwhile saying that the very same results would be obtained if the renormalization conditions were defined in another numerical value of the

external momenta (Weinberg, 1996b). The most general case is the definition of renormalized parameters on a sliding scale μ. To accomplish this renormalization conditions in a point μ the following conditions must be imposed (Gross, 1981):

$$\Gamma^{(2)}(p^2) = -m_\mu^2 \quad \text{at} \quad p^2 = -\mu^2, \tag{38}$$

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = -\lambda_\mu \tag{39}$$

$$\text{at} \quad p_i^2 = \mu^2 \text{ and } p_i p_j = \frac{\mu^2}{3}, i \neq j, \tag{40}$$

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}(p^2) = 1 \quad \text{at} \quad p^2 = \mu^2. \tag{41}$$

The renormalized coupling (38), the renormalized mass (39), and the field normalization (41) are defined at a sliding scale μ.

As an example, consider the one-loop four-point Green calculation

$$\Gamma_1^{(4)} = -z_\lambda \lambda + (z_\lambda \lambda)^2 \left[\int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+p)^2 - m^2](k^2 - m^2)} \right] + O(\lambda^3). \tag{42}$$

We identify the logarithmically divergent integral

$$I = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{[(k+p)^2 - m^2](k^2 - m^2)}. \tag{43}$$

The symbol Λ presupposes an implicit regularization. To separate the logarithmic divergence according to the IRT, one should apply the T^0 operator on (43):

$$I = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} - \int \frac{d^4k}{(2\pi)^4} \frac{p^2 + 2pk}{[(k+p)^2 - m^2](k^2 - m^2)^2}. \tag{44}$$

The first integral is divergent and the second finite

$$I = \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} - \frac{i}{(4\pi)^2} Z_0(m^2, m^2, p^2), \tag{45}$$

where

$$Z_0(m^2, m^2, p^2) = \int_0^1 dz \, \ln\left(\frac{p^2 z(1-z) - m^2}{-m^2} \right). \tag{46}$$

In calculating the finite part of (43) standard methods have been used (Pokorski, 1990). Defining z_λ in order to cancel the divergence and imposing the renormalization conditions (39) on (42) yields the expansion in the conventional coupling

$$\lambda_\mu = -\lambda - \frac{3}{2} \frac{1}{(4\pi)^2} Z_0(\mu^2, m^2, m^2)\lambda^2 + O(\lambda^3). \tag{47}$$

Notice that since no explicit form of a regulator has been used, one can make immediate contact with other regularizations. The remarkable aspect of (47) is

that the dependence on the sliding scale μ of the coupling λ_μ is entirely concentrated on the Z_0 function. In other words, the parameter λ is "fixed" regarding the sliding scale μ. This fact points toward a generalization, viz., that the very physical content of a theory is concentrated in finite parts, which stems from an infinite renormalization procedure. Details of calculations of one-loop quantum electrodynamic amplitudes and their associated Ward identities by using IRT can be found in Battistel *et al.* (1998), Scarpelli *et al.* (2001), and Gobira and Nemes (submitted). In what follow, we present the functions which systematize the finite parts of two- and three-point amplitudes and some useful relations between them in some specific examples.

2.1. The Z_α Functions

The application of the T operator in nth-order Green's function yields finite parts as stated in Section 2. In one-loop calculations, the two-point amplitudes with at least two propagators and one external momenta will be systematized by the dimensionless Z_α functions[5] (Battistel, 1999):

$$Z_\alpha\left(p^2, m_1^2, m_2^2\right) = \int_0^1 dz z^\alpha \ln\left(\frac{p^2 z(1-z) - \left(m_1^2 - m_2^2\right)z - m_1^2}{-m_2^2}\right) \tag{48}$$

where m_i^2 stands for a mass parameters, p^2 the external momentum, and $\alpha \geq 0$. Usually, two-point Green functions are restricted to single mass particles. Taking $m_1^2 = m_2^2 \equiv m^2$, the Z_α functions assume their most simple form

$$Z_\alpha(p^2, m^2, m^2) = \int_0^1 dz z^\alpha \ln\left(\frac{p^2 z(1-z) - m^2}{-m^2}\right) \tag{49}$$

Equation (49) is not restricted to one mass parameter only, since the following identity holds:

$$Z_\alpha(p^2, m^2, M^2) = Z_\alpha(p^2, m^2, m^2) + \frac{1}{\alpha+1} \ln\left(\frac{m^2}{M^2}\right), \tag{50}$$

where M^2 stands for another mass parameter. An important aspect of quantum field theory calculations is the study of Green functions in the asymptotic region (Peskin and Schroeder, 1995). In the limit $p^2 \gg m^2$, (49) becomes

$$\lim_{p^2 \gg m^2} Z_\alpha(p^2, m^2, m^2) \to \frac{1}{1+\alpha} \ln\left(\frac{p^2}{m^2}\right). \tag{51}$$

[5] The external momenta will restrict to the Euclidean region $p^2 < 0$.

Some examples of the use of the Z_α functions are in order. Consider the quantum electrodynamics bare Lagrangian density (Weinberg, 1996a)

$$\mathcal{L} = i\bar{\Psi}_B\gamma^\nu\partial_\nu\Psi_B - m_B\bar{\Psi}_B\Psi_B - \frac{1}{4}F_B^{\alpha\beta}F^B_{\alpha\beta} + e_B\bar{\Psi}_B\gamma^\nu A_\nu^B\Psi_B, \tag{52}$$

where

$$F_B^{\alpha\beta} \equiv \partial^\alpha A_B^\beta - \partial^\beta A_B^\beta \tag{53}$$

and

$$\bar{\Psi}_B \equiv \Psi_B^\dagger\gamma^0. \tag{54}$$

Multiplicative renormalization constants yield renormalized parameters

$$A_\nu^B = \sqrt{z_3}A_\nu, \tag{55}$$

$$\Psi_B = \sqrt{z_2}\Psi, \tag{56}$$

$$e_B = \frac{z_1}{z_2\sqrt{z_3}}e, \tag{57}$$

and

$$m_B = \frac{z_0}{z_2}m. \tag{58}$$

Canonical renormalization conditions define renormalized on-shell parameters[6], viz.

$$\Sigma(\not{p} = m) = 0. \tag{59}$$

$$\frac{d}{d\not{p}}\Sigma(\not{p})\,|_{p=m} = 0, \tag{60}$$

$$\Pi(q^2 = 0) = 0, \tag{61}$$

and

$$-ie\Gamma^\nu(p - q = 0) = -ie\gamma^\nu, \tag{62}$$

where (59) fixes the electron mass m, (60) and (61) fix the residues of the electron and photon propagators at 1 respectively, and (62) fixes the electron charge to be e. Although quantum electrodynamics has a "natural" definition of the parameters e and m, renormalization conditions can be imposed in order to define the parameters on a sliding scale μ.
For instance, define (62) off-shell:

$$-ie\Gamma^\nu(p - q = 0) = -ie\gamma^\nu, \tag{63}$$

[6] Here we use the notation $\not{p} \equiv \gamma_\mu p^\mu$.

where μ stands for a sliding scale. Multiplicative renormalization yields a renormalized Lagrangian density whose parameters were defined in a conventional renormalization point

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \bar{\Psi}[\gamma^{\mu}(\partial_{\mu} + ieA_{\mu}) + m]\Psi. \tag{64}$$

Perturbative calculations on (64) yield one-loop first-order self-energy and vacuum polarization tensor. The first is given by

$$-i\,\Sigma(p) = -e^2 \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{\gamma_{\mu}(\gamma^{\mu} p_{\mu} - \gamma^{\mu} k_{\mu} + m)\gamma^{\mu}}{(2\pi)^4[([p-k)^2 - m^2](k^2 - m^2)}. \tag{65}$$

The use of IRT yields (Baêta Scarpelli *et al.*, 1998)

$$\Sigma(p) = -ie^2(\gamma^{\mu} p_{\mu} - 4m)I_{\ell}(m^2) \tag{66}$$

$$+ \frac{e^2}{8\pi^2}[(\gamma^{\mu} p_{\mu} - 2m)Z_0(\kappa^2, m^2, p^2) + \gamma^{\mu} p_{\mu} Z_1(\kappa^2, m^2, p^2)], \tag{67}$$

where κ is an infrared cutoff, m the electron mass, and p the external momentum. In (67) we separate the amplitude in a basic divergent integral with logarithmic divergence (in the limit $\Lambda \to \infty$:

$$I_{\ell}(m^2) = \int_{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - m^2)^2} \tag{68}$$

and a finite part systematized by two Z_{α} functions

$$Z_0(\mu^2, m^2, p^2) = \int_0^1 dz\, \ln\left(\frac{p^2 z(1-z) + (\mu^2 - m^2) - \mu^2}{-m^2}\right) \tag{69}$$

and

$$Z_1(\mu^2, m^2, p^2) = \int_0^1 dz z\, \ln\left(\frac{p^2 z(1-z) + (\mu^2 - m^2) - \mu^2}{-m^2}\right). \tag{70}$$

Another example is the vacuum polarization tensor

$$-i\,\Pi_{\mu\nu}(q) = -e^2 \int_{\Lambda} \frac{d4k}{(2\pi)^4} \,\mathrm{Tr}\left\{\frac{\gamma_{\nu}(\gamma^{\mu} k_{\mu} - \gamma^{\mu} q_{\mu} + m)\gamma_{\mu}(\gamma^{\mu} k_{\mu} + m)}{[(k-q)^2 - m^2](k^2 - m^2)}\right\}, \tag{71}$$

which yields (Baêta Scarpelli *et al.*, 1998) according to the IRT

$$\Pi_{\mu\nu}(q) = -\frac{4}{3}e^2 \frac{1}{(4\pi)^2}(q_{\mu}q_{\nu} - q^2 g_{\mu\nu})$$

$$\times \left[\frac{1}{q^2}(q^2 + 2m^2)Z_0(m^2, m^2, q^2) + \frac{1}{3}\right] \tag{72}$$

$$-\frac{4}{3}e^2(q_{\mu}q_{\nu} - q^2 g_{\mu\nu})I_{\ell}(m^2), \tag{73}$$

where finite and divergent contributions are clearly separated.

Let us now consider the functions associated with three-point Green's functions.

2.2. The $\xi_{\alpha\beta}$ Functions

We present a set of functions which characterizes three-point functions in one-loop calculations. Their most general form reads (Battistel, 1999)

$$\xi_{\alpha\beta}\left(\mu_1^2, \mu_2^2, \mu_3^2, p^2, q^2\right) = \int_0^1 dz \int_0^{1-z} dy \, \frac{z^\alpha y^\beta}{Q\left(y, z, \mu_1^2, \mu_2^2, \mu_3^2, p^2, q^2\right)} \tag{74}$$

where μ_1^2, μ_2^2, and μ_3^2 stand for mass parameters, p^2, q^2 external momenta, and $\alpha, \beta \geq 0$. Q is defined as

$$Q\left(y, z, \mu_1^2, \mu_2^2, \mu_3^2, p^2, q^2\right) = p^2 y(1-y) + \left(\mu_1^2 - \mu_2^2\right)y + q^2 y(1-y) \tag{75}$$

$$+ \left(\mu_1^2 - \mu_3^2\right)z - \mu_1^2 - 2(pq)yz. \tag{76}$$

When $\alpha = \beta = 0$, (74) reduces to a Spence function (Pokorski, 1990). Also, except for $\alpha = \beta = 0$, the $\xi_{\alpha\beta}$ functions can be reduced to the Z_α functions (Battistel, 1999). For instance, the following identities involving Z_α and $\xi_{\alpha\beta}$ functions are very useful in proving Ward identities (Gobira *et al.*, 2000):

$$q^2\xi_{10} + pq\xi_{01} = \frac{1}{2}\Big[-Z_0\left(\mu_1^2, \mu_2^2, p^2\right) \tag{77}$$

$$- Z_0\left(\mu_2^2, \mu_2^2, (p-q)^2\right) \tag{78}$$

$$- \left(q^2 + \mu_1^2 - \mu_2^2\right)\xi_{00}\Big], \tag{79}$$

$$q^2\xi_{20} + pq\xi_{11} = \frac{1}{2}\left[\frac{1}{2}Z_0\left(\mu_2^2, \mu_2^2, (p-q)^2\right) - Z_1\left(\mu_2^2, \mu_2^2, (p-q)^2\right)\right. \tag{80}$$

$$+ \frac{3}{2}\left(q^2 + \mu_1^2 - \mu_2^2\right)\xi_{10} + \frac{1}{2}\left(p^2 + \mu_1^2 - \mu_2^2\right)\xi_{01} \tag{81}$$

$$\left. - \left(\frac{1}{2} + \mu_1^2\xi_{00}\right)\right], \tag{82}$$

$$q^2\xi_{11} + pq\xi_{02} = \frac{1}{2}\left[Z_1\left(\mu_2^2, \mu_2^2, (p-q)^2\right) - Z_0\left(\mu_2^2, \mu_2^2, (p-q)^2\right)\right. \tag{83}$$

$$\left. + Z_1\left(\mu_1^2, \mu_2^2, p^2\right) + \frac{1}{2}\left(q^2 + \mu_1^2 - \mu_2^2\right)\xi_{01}\right]. \tag{84}$$

The asymptotic limit of the $\xi_{\alpha\beta}$ functions can be calculated with the help of (51) and with the asymptotic limit of the Spence function:

As an example, consider the one-loop correction for the quantum electrodynamics vertex

$$-ie\Gamma_\nu(p,q) = -e^3 \int_\Lambda \frac{d^4k}{(2\pi)^4} \frac{\gamma_\mu(\gamma^\mu p_\mu - \gamma^\mu k_\mu + m)\gamma_\nu(\gamma^\mu q_\mu - \gamma^\mu k_\mu + m)\gamma^\mu}{[(p-k)^2 - m^2][(q-k)^2 - m^2](k^2 - \mu^2)} \tag{85}$$

where p and q are external momenta and μ an infrared cutoff. Using the IRT, one gets (Baêta Scarpelli *et al.*, 1998)

$$-ie\Gamma_\nu(p,q) = e^3 \frac{-i}{(4\pi)^2}[4m(p_\nu + q_\nu) - \gamma_\nu(p^2 + q^2) - 2\gamma^\mu p_\mu \gamma_\nu \gamma^\beta q_\beta]\xi_{00} \tag{86}$$

$$+ e^3 \frac{i8m}{(4\pi)^2}(p_\nu \xi_{10} + q_\nu \xi_{01}) \tag{87}$$

$$- e^3 \frac{i}{(4\pi)^2}[\gamma_\nu(p^\mu + q^\mu) - 2(q_\beta \gamma^\beta \gamma^\mu \gamma_\nu + p_\delta \gamma^\delta \gamma_\nu \gamma^\mu)]$$

$$\times (p_\mu \xi_{10} + q_\mu \xi_{01}) \tag{88}$$

$$+ e^3 \frac{i}{(4\pi)^2}[2\gamma_\nu F + 4\gamma^\mu(p_\mu p_\nu \xi_{20} + q_\mu q_\nu \xi_{02}$$

$$+ (p_\mu q_\nu + p_\nu q_\mu)\xi_{11}] \tag{89}$$

$$+ e^3 \frac{i}{(4\pi)^2}[Z_0(\kappa^2, m^2, p^2) + Z_0(\kappa^2, m^2, q^2)]$$

$$- \gamma_\nu e^3 I_\ell(m^2) \tag{90}$$

and

$$\xi_{\alpha\beta} = \xi_{\alpha\beta}(\kappa^2, m^2, p^2, q^2), \tag{91}$$

$$F = F(\kappa^2, m^2, p^2, q^2) \equiv \int_0^1 dz \int_0^{1-z} dy \, \ln\left(\frac{Q(y, z, \kappa^2, m^2, p^2, q^2)}{-m^2}\right), \tag{92}$$

and Q defined by (76). With the aid of (79), (82), and (84) the one-loop (90) finite parts could be reduced to Z_α and ξ_{00} functions. In the next section we will briefly review some aspects of renormalization group equation and show how a mass scale enters in the definition of renormalized parameters when IRT is used.

3. THE RENORMALIZATION GROUP AND THE IMPLICIT REGULARIZATION TECHNIQUE

The renormalization group (RG) techniques were originally introduced by Gell-Mann and Low (1954) as a way of dealing with the large logarithms that may break down perturbation procedures (Weinberg, 1996b). Let $\Gamma(E, g, m)$ be a physical amplitude that depends on an overall energy scale E, a dimensionless

coupling constant g, and a mass called m. If Γ has dimensionality $[mass]^D$ then simple dimensional analysis tells us that

$$\Gamma(E, g, m) = E^D \Gamma\left(1, g, \frac{m}{E}\right). \tag{93}$$

In the limit $E \to \infty$, we might expect the simple power behavior

$$\Gamma(E, g, m) \to E^D \Gamma(1, g, 0). \tag{94}$$

Instead of this simple power behavior, in perturbation calculations the factor E^D is found to be accompanied by powers of $\ln(E/m)$, which can enter as $E \to \infty$ with fixed m only if Γ, at fixed E, becomes singular as $m \to 0$.

Consider now a physical amplitude $\Gamma(E, g_\mu, m_\mu, \mu)$ that depends on dimensionless coupling g_μ and mass m_μ, defined *by renormalization conditions* on a sliding energy scale μ. We define g_μ in such a way that, at least for

$$\mu \gg m_\mu, \tag{95}$$

g_μ has no dependence on the scale m_μ of the mass of the theory. Again simple dimensional analysis tell us that

$$\Gamma(E, g, m, \mu) = E^D \Gamma\left(1, g_\mu, \frac{m_\mu}{E}, \frac{\mu}{E}\right). \tag{96}$$

Since μ is arbitrary, we can choose $\mu = E$. Thus

$$\Gamma(E, g, m, \mu) = E^D \Gamma(1, g_E, 0, 1). \tag{97}$$

This has no zero mass singularities because, by construction, g_E does not depend on m_μ for $E \gg m_\mu$ and so there are no large logarithms, and we can use perturbation theory to calculate Γ in terms of g_E as long as g_E itself remains small.[7] Consider Eq. (96), an n-point amputated Green's function

$$\Gamma(E, g, m, \mu) \equiv \Gamma_\mu^{(n)}(p_1, \ldots, p_n, m_\mu, g_\mu, \mu) \tag{98}$$

obtained from a bare n-point amputated Green's function via multiplicative renormalization

$$\Gamma_B^{(n)}(p_1, \ldots, p_n, m_B, g_B, \Lambda) = Z_{\phi_\mu}^{-n/2}\left(\frac{\Lambda}{\mu}, \frac{m_\mu}{\mu}, g_\mu\right)$$
$$\times \Gamma_\mu^{(n)}(p_1, \ldots, p_2, m_\mu, g_\mu, \mu), \tag{99}$$

where $p_1, \ldots, p_n$ stand for n external momenta and Λ for an ultraviolet cutoff. Imposing invariance of the LHS of (99) with respect to μ, one gets the RG equation (Gell-Mann and Low, 1954; Stueckenburg and Peterman, 1953)

$$\left(\mu \frac{\partial}{\partial \mu} + \beta_\mu \frac{\partial}{\partial g_\mu} + \gamma_{m_\mu} \frac{\partial}{\partial m_\mu} - n\gamma_\mu\right) \Gamma_\mu^{(n)} = 0, \tag{100}$$

[7] g_E is the finite coupling defined in a conventional renormalization point.

where

$$\beta_\mu \equiv \mu \frac{\partial g_\mu}{\partial \mu}, \tag{101}$$

$$\gamma_{m_\mu} \equiv \mu \frac{\partial m_\mu}{\partial \mu}, \tag{102}$$

and

$$\gamma_\mu \equiv \frac{\mu}{2} \frac{d}{d\mu} \ln(Z_{\phi_\mu}). \tag{103}$$

As stated in the Introduction, in order to properly define the parameters of a theory, one must specify renormalization conditions. These conditions define the values of the Green's functions and, in the process, remove all ultraviolet divergences. But the values of the renormalized parameters are defined in *specific external momenta*. This simple fact is fundamental to understand how an arbitrary mass scale enters in the definition of renormalized parameters when handling amplitudes using the IRT. According to this technique the (nonpolynomial) external momenta dependence of the amplitude is contained in its finite parts, duly separated in terms of the Z_α and/or $\xi_{\alpha\beta}$ functions. Thus, in dealing with Γ using IRT we find that the one-loop renormalized parameters g_μ, m_μ, and ϕ_μ are given in terms of the Z_α and/or $\xi_{\alpha\beta}$ functions, which have explicit dependence on external momenta. Hence the renormalization group coefficients (101), (62), and (96) can be directly obtained from derivatives of the systematized Z_α and/or $\xi_{\alpha\beta}$ functions relative to the external momenta.

Consider the $\lambda\varphi_4^4$ coupling (47) evaluated in a sliding scale μ

$$\lambda_\mu = -\lambda - \frac{3}{2} \frac{1}{(4\pi)^2} Z_0(m^2, m^2, \mu^2)\lambda^2 + O(\lambda^3). \tag{104}$$

The β-function (101) can be evaluated directly from its definition

$$\beta_\mu \equiv \mu \frac{\partial}{\partial \mu} \left[-\lambda - \frac{3}{2} \frac{1}{(4\pi)^2} Z_0(m^2, m^2, \mu^2)\lambda^2 + O(\lambda^3) \right]. \tag{105}$$

Notice that the μ-dependence of the coupling resides *only* on the Z_0 function. Hence

$$\beta_\mu = \frac{3}{2} \frac{1}{(4\pi)^2} \left[\int_0^1 dz \, \frac{\mu^2 z(1-z)}{\mu^2 z(1-z) - m_\mu^2} \right] \lambda^2 + O(\lambda^3). \tag{106}$$

Note that the result (106) belongs to the nonasymptotic region. Taking the limit $\mu \gg m_\mu$ yields

$$\beta_\mu = \frac{3}{16\pi^2} \lambda^2 + O(\lambda^3), \tag{107}$$

which is the standard one-loop result to $\lambda\varphi_4^4$ theory.

The same lines of reasoning apply to quantum electrodynamics. The β-function calculation could be performed from its definition (101), with the coupling given by (90). This would lead to quite lengthy calculations, fortunately avoidable by the Ward identity

$$z_1 = z_2, \tag{108}$$

which implies

$$e_B = \frac{e}{\sqrt{z_3}} \tag{109}$$

or, in terms of the fine-structure constant α,

$$\alpha_B \equiv \frac{\alpha}{z_3}. \tag{110}$$

Imposing μ invariance on Eq. (110) yields

$$\beta = \frac{1}{z_3}\mu^2 \frac{\partial z_3}{\partial \mu^2}\alpha + O(\alpha^2). \tag{111}$$

By the IRT the z_3 finite part reads (Baêta Scarpelli *et al.*, 1998)

$$(z_3)_{\text{fin}} = 1 - \frac{\alpha}{3\pi}\left[\left(1 + \frac{2m}{q^2}\right) Z_0(m^2, m^2q^2) + \frac{1}{3}\right], \tag{112}$$

and imposing (63), (112) is evaluated in the external momentum $q^2 = \mu^2$. Again, that is how the sliding scale μ enters in the IRT. In other words, in the IRT, the sliding scale μ is directly related to the renormalization condition. A straightforward calculation yields the well-known one-loop asymptotic quantum electrodynamics β-function[8]

$$\beta = \frac{2}{3\pi}\alpha^2 + O(\alpha^3) \tag{113}$$

4. THE ASYMPTOTIC REGION AND CONNECTION BETWEEN SUBTRACTION SCHEMES

In perturbation theory, the dependence of Green functions on massive parameters is expressed by two differential equations. First, the Callan–Symanzik (CS) equation that describes the breaking of the dilatational invariance under rescaling in the momenta (Gross, 1981):

$$\left(m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial \mu} + \beta\frac{\partial}{\partial g} - n\gamma\right)\Gamma^{(n)}(\varphi) = \alpha \int [-m^2\varphi^2]_2 \Gamma^{(n)}(\varphi), \tag{114}$$

[8] This known result is valid in the region $\mu \gg m$.

where m and μ are mass parameters, g the coupling, and γ the anomalous dimension. The other equation is the RG equation derived above [Eq. (100)]:

$$\left(\mu\frac{\partial}{\partial\mu}+\beta\frac{\partial}{\partial g}+\gamma_m\frac{\partial}{\partial m}-n\gamma\right)\Gamma^{(n)}(\varphi)=0. \tag{115}$$

As we showed, (115) describes the invariance of Green functions under the RG transformations. The original ideas that lead to (115) stem from Stueckelberg and Petermann (1953) and Gell-Mann and Low (1954). Mass independent β-functions in massive theories indicate the fact that the RG transformations are restricted to the asymptotic region (Kraus, 1994). The condition for mass independence in the asymptotic region is the existence of a CS equation of the same form of the RG equation, i.e., the known massless differential operators of the CS equation must be the same operators of the RG equation, evaluated in the large momenta region. It was proved for the massive $\lambda\varphi^4$ theory (though the proof is quite general (Kraus, 1994)) that the minimal subtraction (MS), modified minimal subtraction $\overline{\text{MS}}$ (Collins, 1984), and the Bogolubov Parasiuk Hepp Zimmermann Lowenstein (Zimmermann, 1968, 1969) schemes have normalization properties in the asymptotic region:

$$\lim_{p^2\gg m^2}\frac{\partial}{\partial p^2}\Gamma^{(2)}(p^2=\mu^2)=\sum_{n=0}^{\infty}a^n_{(2)n}\lambda^n \tag{116}$$

$$\lim_{p^2\gg m^2}\Gamma^{(4)}\left(p_i^2=\mu^2,\,p_i\,p_j=-\frac{\mu^2}{3}\right)=\sum_{n=0}^{\infty}a^n_{(4)n}\lambda^{n+1}, \tag{117}$$

where the $a^n_{(k)n}$ are mass independent coefficients. Hence, in all these three schemes the β-functions and the γ-functions of the CS equation and RG equation are the same and mass independent. As we saw above, the result (106) is in the nonasymptotic region and in the limit $\mu\gg m_\mu$, (104) yields using (51)

$$\lambda_\mu=-\lambda+\left[\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(\frac{\mu^2}{m^2}\right)\right]\lambda^2+O(\lambda^3). \tag{118}$$

It is interesting to compare (118) to general one-loop results of MS, $\overline{\text{MS}}$, and the BPHZL schemes (Kraus, 1994):

$$\text{MS:}\quad \lambda_\mu=-\lambda+\left[Z_0(\kappa^2,m^2)+\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(\frac{\mu^2}{m^2}\right)+\ln 4\pi-\lambda_E\right]\times\lambda^2+O(\lambda^3), \tag{119}$$

$$\overline{\text{MS}}\text{:}\quad \lambda_\mu=-\lambda+\left[Z_0(\kappa^2,m^2)+\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right]\lambda^2+O(\lambda^3), \tag{120}$$

$$\text{BPHZL:}\quad \lambda_\mu = -\lambda + \left[Z_0(\kappa^2, m^2) + \frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(-\frac{4\mu^2}{3m^2}\right)\right] \times \lambda^2 + O(\lambda^3). \tag{121}$$

here we denote the renormalization point according to the general convention μ and $\bar{\mu}$ respectively. It is important to observe that the renormalization conditions that yield the above expansions in the finite conventional coupling λ are different from the one we adopted. In (119), (120), and (121), the parameter λ is evaluated in the Euclidean symmetric point ($p^2 < 0$):

$$p_i^2 = \kappa^2 \tag{122}$$

and

$$p_i\, p_j = -\frac{\kappa^2}{3}. \tag{123}$$

As stated above, the choice of this point does not have physical relevant consequences. Anyway, contact with the results (119), (120), and (121) can be done with the help of the identity (50). Taking $\kappa^2 = 0$ in (119), (120), and (121) yields

$$\text{MS:}\quad \lambda_\mu = -\lambda + \left[\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(\frac{\mu^2}{m^2}\right) + \ln 4\pi - \gamma_E\right]\gamma^2 + O(\lambda^3), \tag{124}$$

$$\overline{\text{MS}}\text{:}\quad \lambda_\mu = -\lambda + \left[\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(\frac{\bar{\mu}^2}{m^2}\right)\right]\lambda^2 + O(\lambda^3), \tag{125}$$

$$\text{BPHZL:}\quad \lambda_\mu = -\lambda + \left[\frac{1}{(4\pi)^2}\frac{3}{2}\ln\left(-\frac{4\mu^2}{3m^2}\right)\right]\lambda^2 + O(\lambda^3). \tag{126}$$

and comparing these results with (118) we see indeed that the IRT applied to the $\lambda\varphi^4$ yields the same asymptotic expressions as the above known schemes. Again, we would like to stress that the connection between sliding scales and renormalized parameters is quite natural in the IRT, since it is realized by imposing renormalization conditions.

5. RELATIONSHIP BETWEEN DIMENSIONAL, DIFFERENTIAL, AND IMPLICIT RENORMALIZATIONS

In this section we will show how DR, DFR (differential regularization) (Dunne, 1992), and IRT are related regarding the appearance of a renormalization scale. Such comparison is interesting since DR is widely used for analyzing renormalizable QFT (particularly those involving gauge symmetry) whereas DFR is an elegant framework which, as well as IRT, does not recourse to analytical

continuation on the space–time dimension. The idea behind DFR is to redefine products of Green's functions in the Euclidean (position) space as proper distributions: the singularities at coincident points (which have no Fourier transform) are expressed as derivatives of less singular terms (which do have Fourier transforms). We follow Dunne (1992).

Consider the identity

$$|x|^{-p} = \frac{\Box |x|^{-p+2}}{(-p+2)(d-p)}. \tag{127}$$

For $p = d$, we cannot use (127) because of the pole. According to the DFR rules, we must instead substitute

$$|x|^{-p}\ |_{\text{reg}} := \frac{1}{2(2-d)} \Box \frac{\ln M^2 |x|^2}{|x|^{d-2}}, \tag{128}$$

which holds when $|x| \neq 0$ and the dependence on an arbitrary mass scale M appears for dimensional reasons. It plays the role of scale in the Callan–Symanzik RG equation.

To make contact with RD we can use identity (127) by extending d to $d - r\epsilon$ where it is well defined to write

$$\begin{aligned} \mu^{r\epsilon} |x|^{-d+r\epsilon} &= \frac{1}{\epsilon} \mu^{r\epsilon} \frac{1}{r(2-d+r\epsilon)} \Box |x|^{d-2}\ \ln M^2 |x|^{-d+r\epsilon+2} \\ &= \frac{1}{\epsilon} \frac{4\pi^{d/2}}{r(2-d+r\epsilon)\Gamma(d/2-1)} \delta^{(d)}(x) \\ &\quad + \frac{1}{2(2-d)} \Box \frac{\ln \mu^2 |x|^2}{|x|^{d-2}} + O(\epsilon). \end{aligned} \tag{129}$$

Now we can clearly see that finite (no counterterms) part of (129) is identical to DFR after subtracting the infinite and a finite $O(\epsilon^0)$ counterterms represented by the delta function and identifying μ with M.

As a matter of illustration consider the one-loop four-point function of φ^4 theory. In DR it reads

$$\Gamma(p^2, m^2) = \frac{i\lambda^2 \mu^\epsilon}{16\pi^2 \epsilon} - \frac{i\lambda^2 \mu^\epsilon}{32\pi^2} \left\{ \gamma + \int_0^1 dz\ \ln \left[\frac{p^2 z(1-z) - m^2}{4\pi \mu^2} \right] \right\}. \tag{130}$$

Defining counterterms to subtract the pole and the term proportional to γ enables us to write

$$\Gamma^R_{\text{RD}}(p^2, m^2) = \frac{i\lambda^2}{32\pi^2} \left\{ \ln \left(\frac{m^2}{4\pi \mu^2} \right) - Z_0(p^2, m^2) \right\}. \tag{131}$$

According to the DFR rules, $\Gamma^R(p^2, m^2)$ is written as (Haagensen and Latorre, 1992)

$$\Gamma^R_{\rm DFR}(p^2, m^2) = \frac{i\lambda^2}{32\pi^2}\left\{\ln\left(\frac{m^2}{M^2}\right) - Z_0(p^2, m^2)\right\}, \tag{132}$$

from which is clear the equivalence of (131) and (132) identifying $M^2 = 4\pi\mu^2$. Using now the IRT rules we can write

$$\Gamma^R_{\rm IRT}(p^2, m^2) = \frac{\lambda^2}{2}\left\{I_\ell(m^2) - \frac{i}{4\pi^2}Z_0(p^2, m^2)\right\}. \tag{133}$$

Taking

$$I_\ell(m^2) = I_\ell(\eta^2) + \frac{i}{4\pi^2}\ln\frac{\eta^2}{m^2}, \tag{134}$$

where the second term on the RHS of the above equation parametrizes a finite arbitrary counterterms, into (133) and defining a counterterms to subtract $I_\ell(\eta^2)$ leads to

$$\Gamma^R_{\rm IRT}(p^2, m^2) = \frac{i\lambda^2}{32\pi^2}\left\{\ln\left(\frac{m^2}{\eta^2}\right) - Z_0(p^2, m^2)\right\}, \tag{135}$$

making clear the connection between these three schemes.

6. CONCLUSIONS

In treating quantum field theory amplitudes perturbatively, a renormalization procedure must be imposed to define, order by order, the parameters of the theory. Such perturbative procedures are plagued by divergences. To remove the divergences and, in the process, redefine the parameters of the theory, a subtraction scheme associated with a regularization method must be employed. This procedure cannot be performed in a unique way since divergences are present. Thus any subtraction algorithm must carry a parameter to accomplish the arbitrariness of this infinite renormalization. A nontrivial issue is what is the role played by this parameter in the theory. The most general prescription is to state renormalization conditions which define the values of the Green's functions in a sliding scale and, in the process, remove all ultraviolet divergences. When the dimensional regularization is used, an arbitrary mass parameter must be introduced in order to keep the coupling dimensionality. This parameter is linked nontrivially to the parameter introduced via renormalization conditions and thus it is used for renormalization group purposes. Employing the BPHZL scheme, no use of an explicitly cutoff is required a priori. But the for practical purposes, very often the dimensional regularization is implemented in intermediary steps and hence breaking the 4-space dimensionality. Moreover, in IRT we explicitly construct the counterterms without

changing the structure of the integrand. Using the IRT the role of renormalization conditions concerning the introduction of a sliding scale is straightforward: the arbitrariness of the sliding scale is directly related to the unavoidable arbitrariness in the process of separating finite and divergent contributions to physical amplitudes. Moreover, as we have shown, the finite one-loop parts are systematized in a small set of functions, namely the Z_α and $\xi_{\alpha\beta}$ functions. The renormalization conditions act directly in the Z_α and $\xi_{\alpha\beta}$ functions which contain all the arbitrariness of the process. The connection of the systematized functions with renormalization group results shows that IRT results are calculated in nonasymptotic region and agree with standard calculations in the large momenta region. The extension of our approach to non-abelian gauge theories where BPHZ fails to preserve the relevant Slavnov–Taylor identities (also in the finite part) is presently under study. Since we do not change the structure of the integrand and keep the arbitrariness expressed by differences between divergent integrals to be fixed on physical grounds related to momentum routing invariance, we expect that we can fully preserve gauge invariance.

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